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Stabilizing nonlinear dynamical systems by an adaptive adjustment mechanism

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An adaptive adjustment mechanism is applied to stabilize multidimensional dynamical systems. Without utilizing any prior knowledge of the system itself, nor extra external control signals, the mechanism can ensure a large class of chaotic systems to converge to their ''generic'' stable periodic orbits.

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MOTIVATIONS

Much attention has been focused on stabilizing chaotic dynamical systems in recent years [1,2], and various algorithms have been designed to achieve such goal. Most algorithms, however, either require *a priori* knowledge about the system, or force the system to converge to the periodic orbits that are biased from the original system. One exception is the *adaptive adjustment mechanism* (AAM), which utilizes neither prior knowledge of the system itself nor extra external control signals, and forces all one-dimensional discrete systems to converge to their original periodic orbits. This Rapid Communication applies the same principle to the much more complicated multidimensional dynamical systems, and explores the pros and cons.

Consider an n-dimensional nonlinear discrete system defined by

$$X(t+1)=F(X(t)), \tag{1}$$

where $X=(x_1,x_2,\ldots,x_n)$, and $F=(f_1,f_2,\ldots,f_n)$, with f_i being well defined functions on a domain $\mathcal{D}$.

By *adaptive adjustment mechanism*, we mean the following modified system:

$$X(t+1)=\tilde{F}(X(t)):=(1-\gamma)F(X(t))+\gamma X(t), \tag{2}$$

where γ is positive controlling parameter, and is referred to as *adaptive parameter* hereafter. The practical implementation is illustrated in Fig. 1. Expressing Eq. (2) as $X(t+1)=F(X(t))+\gamma X(t)-F(X(t))$, we see that AAM forces a feedback adjustment whenever any variable strays away from its previous state. It is easy to verify that the system with AAM processes the following properties:

Corollary 1: *The systems F and $\tilde{F}$ share exactly the same set of fixed points.* (Proof omitted.)

Corollary 1 implies the fixed points of AAM are ''generic'' in the sense that they are inherited directly from the original system. Denote $DF(\overline{X})$ as the Jacobian matrix of the original system F evaluated at $\overline{X}$ with $\{\lambda_1,\lambda_2,\ldots\lambda_n\}$ as the n roots of the characteristic equation, i.e.,

$$|\lambda\mathbf{I}-DF(\overline{X})|=\prod_{j=1}^{n}(\lambda-\lambda_j)=0, \tag{3}$$

where $\mathbf{I}$ is a unit matrix. And let $D\tilde{F}(\overline{X})$ be the Jacobian matrix of the system $\tilde{F}$ evaluated at $\overline{X}$ and $\{\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots\tilde{\lambda}_n\}$ be the related eigenvalues, so that

$$|\lambda\mathbf{I}-D\tilde{F}(\overline{X})|=\prod_{j=1}^{n}(\lambda-\tilde{\lambda}_j)=0. \tag{4}$$

Then we have

Corollary 2: *For each and every fixed point of F and $\tilde{F}$, there exists the following one-to-one correspondence between their eigenvalues:*

$$\tilde{\lambda}_j=(1-\gamma)\lambda_j+\gamma, \quad j=1,2,\ldots,n. \tag{5}$$

Proof:
It follows from Eq. (2) that

$$D\tilde{F}=(1-\gamma)DF+\gamma\mathbf{I}. \tag{6}$$

The characteristic equation under the map $\tilde{F}$ is given by

$$\begin{aligned}|\tilde{\lambda}\mathbf{I}-D\tilde{F}(\overline{X})|&=|(\tilde{\lambda}-\gamma)\mathbf{I}-(1-\gamma)DF|\\&=(1-\gamma)^n|\hat{\lambda}\mathbf{I}-DF(\overline{X})|\\&=(1-\gamma)^n\prod_{j=1}^{n}(\hat{\lambda}-\lambda_j),\end{aligned}$$

$$\text{where}\quad \hat{\lambda}:=\frac{\tilde{\lambda}-\gamma}{1-\gamma}, \tag{7}$$

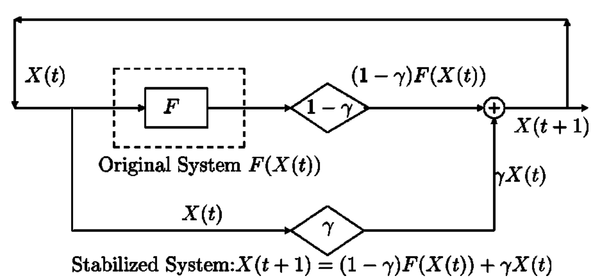

FIG. 1. Adaptive adjustment mechanism.

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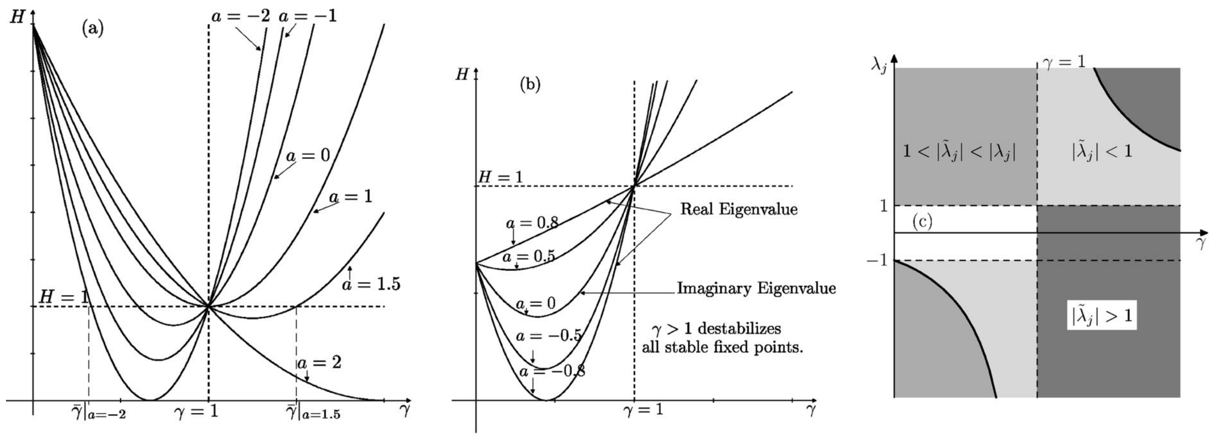

FIG. 2. Effects of γ.

so that $\hat{\lambda}=\lambda_j$ would imply identity (5). Q.E.D.

A fixed point of $\tilde{F}$ is stable if and only if $max|\lambda_j|<1$. Corollary 1 and 2 together enable us to adjust the eigenvalues by suitable choice of an adaptive parameter γ only.

To grasp a general picture of the effect of introducing AAM, we consider the situation in which an eigenvalue is complex. Now supposing that λ_1 and λ_2 are a pair of complex conjugates such that

$$\lambda_{1,2}=a\pm b\mathbf{i}, \tag{8}$$

and the modulus is given by $|\lambda_{1,2}|=\sqrt{a^2+b^2}$.

The eigenvalues corresponding to the $\tilde{F}$ are given by

$$\tilde{\lambda}_{1,2}=(1-\gamma)(a\pm b\mathbf{i})+\gamma=[(1-\gamma)a+\gamma]\pm(1-\gamma)b\mathbf{i}, \tag{9}$$

with the modulus

$$|\tilde{\lambda}_{1,2}|=\sqrt{H(\gamma)}=\sqrt{((1-\gamma)a+\gamma)^2+(1-\gamma)^2b^2},$$
$$\text{where}\quad H(\gamma)\hat{=}((1-\gamma)a+\gamma)^2+(1-\gamma)^2b^2. \tag{10}$$

Let $\bar{\gamma}$ be the critical adaptive parameter such that $H(\bar{\gamma})=1$. Solving from Eq. (10), we have

$$\bar{\gamma}=1+\frac{2(a-1)}{(a-1)^2+b^2}, \tag{11}$$

$$\text{with}\quad H'(\bar{\gamma})=2(a-1). \tag{12}$$

Therefore, $\bar{\gamma}\lesseqgtr 1$ and $H'(\bar{\gamma})\lesseqgtr 0$ if and only if $a\lesseqgtr 1$.

For the special cases $\gamma=0$ (without AAM) and $\gamma=1$ (no effect of the original map), there exist the following identities and inequalities:

$$H(0)=a^2+b^2>0, \tag{13}$$

$$H'(0)=2(a-H(0)), \tag{14}$$

$$H(1)=1, \tag{15}$$

$$H'(1)=2(1-a)\lesseqgtr 0\quad\text{if}\quad a\lesseqgtr 0. \tag{16}$$

Also note that

$$H''(\gamma)=2((1-a)^2+b^2)>0. \tag{17}$$

These relationships enable us to explore how $H(\gamma)$ is changed with γ in terms of the nature of $H_j(0)$:

Case A. $H(0)>1$ (and hence $|\lambda_{1,2}|>1$):

If $a<1$, we have $H'(0)<0$ and $H'(1)>0$. The identity $H(1)=1$ implies that there always exists a $\bar{\gamma}$ such that $H(\gamma)<1$ for all $\gamma\in(\bar{\gamma},1)$. However, when $\gamma>1$, $H(\gamma)$ will resume to exceed unity.

If $a>1$, both $H'(0)<0$ and $H'(1)<0$ hold true. The convex property of H reveals that H is decreasing along the increase of γ from 0 to 1 (but never to the extent that it is less than unity). Therefore, there always exists a $\bar{\gamma}$ such that $H(\gamma)<1$ for all $\gamma\in(1,\bar{\gamma})$. But when $\gamma>\bar{\gamma}$, $H(\gamma)$ starts to exceed unity again.

If $a=1$, although $H'(\gamma)<0$ holds for $\gamma<1$, $H(\gamma)$ is always greater than unity.

The above analysis is illustrated in Fig. 2(a), where $H(\gamma)$ is plotted against γ with the assumption of $H(0)\equiv 4$. We see, *no matter what a is, introducing AAM with $\gamma<1$ always helps in reducing the magnitude of the modulus*. It also observed that the modulus of an imaginary eigenvalue ($a=0,b>1$) can only be reduced by a γ that is less than unity.

Case B. $H(0)<1$ (and hence $|\lambda_{1,2}|<1$): This case exists only when $a<1$. Since $H'(0)=2(a-H(0))$, introducing a γ that is less than unity may decrease or increase the eigenvalue at the beginning, but finally increase again until $H(1)=1$. Hence, when $\gamma\in(0,1)$, $H(\gamma)$ will never exceed unity so that the stability of the periodic orbit is preserved. To the contrary, when $\gamma>1$, $H(\gamma)$ will become greater than unity so as to destabilize a stable periodic orbit. Case B is illustrated in Fig. 2(b), where $H(0)\equiv 0.6$ is set.

When the original eigenvalue is real ($\lambda_j=a$), which is always true for a one-dimensional system ($n=1$), Eq. (11) is simplified to

$$\bar{\gamma}=\frac{(\lambda_j+1)}{\lambda_j-1}. \tag{18}$$

A more detailed illustration for the relationship between λ and $\tilde{\lambda}$ with respect to γ for real eigenvalues is demonstrated in Fig. 2(c).

The following conclusion follows from the above analysis:

Theorem 3: *For an n-dimensional dynamical system defined in Eq. (1), supposing that $\overline{X}$ is a fixed point of F and a_j is the real part of the eigenvalue λ_j, for j=1,2, . . . ,n, then*

Case I. if $a_j<1$ for all $j=1,2,\ldots,n$, there exists a $\overline{\gamma}$ such that, for all $\gamma\in(\overline{\gamma},1)$, all respective modulus under AAM can be reduced to a magnitude that is less than unity, and hence, the original unstable fixed point will be stabilized;

Case II. if $a_j>1$ for all $j=1,2,\ldots,n$, there exists a $\overline{\gamma}$ such that, for all $\gamma\in(1,\overline{\gamma})$, all respective modulus under AAM can be reduced to a magnitude that is less than unity, and hence, the original unstable fixed point is stabilized;

Case III. if some a_j's are greater than unity, but others are less than unity, then the unstable fixed point cannot be stabilized by simple AAM defined by Eq. (2).

For a one-dimensional dynamical system, case III does not occur, therefore, all unstable periodic points can be stabilized by AAM. Geometrically, adopting an adaptive parameter γ that is less than unity can stabilize all fixed points with down-sloping branches ($\lambda<0$), while fixed points with upper-sloping branches ($\lambda>1$) can be stabilized by an adaptive parameter γ that is greater than unity [3]. Actually, when $n=1$, Eq. (2) reduces to

$$\tilde{f}(x(t))=(1-\gamma)f(x(t))+\gamma x(t), \tag{19}$$

and hence

$$\tilde{f}'(x)=(1-\gamma)f'(x)+\gamma, \tag{20}$$

$$\tilde{f}''(x)=(1-\gamma)f''(x). \tag{21}$$

Therefore, if $\gamma<1$, $\tilde{f}''(x)$ preserves the sign of $f''(x)$, and from the discussion above, in most cases, $\tilde{f}'$ has the same sign of $f'(x)$. The net effect of introducing γ is to ''squeeze'' the original system toward its diagonal axis in the phase diagram.

However, if $\gamma>1$, in most cases, $\tilde{f}'$ possesses the opposite sign of $f'(x)$, while $\tilde{f}''(x)$ is always opposite to $f''(x)$. The net effect of introducing γ is to reflect the original system against the diagonal line in the phase diagram.

The implementation of AAM in one-dimensional discrete dynamics (19) enjoys a special property when $\gamma<1$, *preserving the domain (fluctuation range) of the original process f*. Obviously, if the domain of a chaotic process f is given by $I=(x_{min},x_{max})$, with $x_{min}<x_{max}$, which are achieved by f at x^l and x^h, respectively, i.e., $x_{min}=f(x^l)$ and $x_{max}=f(x^h)$, with $x_{min}\leqslant x^l<x^h\leqslant x_{max}$, then

$$\begin{aligned}\tilde{f}(x^l)&=(1-\gamma)f(x^l)+\gamma x^l\\&=(1-\gamma)x_{min}+\gamma x^l\geqslant(1-\gamma)x_{min}+x_{min}=x_{min},\end{aligned}$$

$$\begin{aligned}\tilde{f}(x^h)&=(1-\gamma)f(x^h)+\gamma x^h\\&=(1-\gamma)x_{max}+\gamma x^h\leqslant(1-\gamma)x_{max}+x_{max}=x_{max},\end{aligned}$$

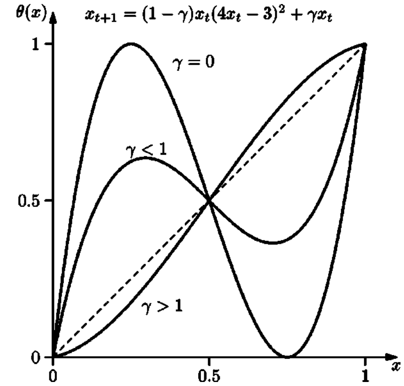

FIG. 3. Example of the cubic map.

that is, $\tilde{f}$ maps I into I itself.

But this property does not hold when $\gamma>1$ and when $n>1$. Figure 3 provides an illustration with the cubic map defined by $x_{t+1}\hat{=}x_t(4x_t-3)^2$, from which we see that the fixed point $\overline{x}_2=\frac{1}{2}$ is stabilized with $\gamma=\frac{1}{2}$, while the fixed point at two ends $x_1=0$ and $x_3=1$ are stabilized by $\gamma\in[1,\gamma_{\max}]$, where $\gamma_{\max}=[\theta'(0)+1/\theta'(0)-1]=\frac{5}{4}$. However, no matter what value γ may take, the AAM always preserves the positions of these fixed points.

NUMERICAL SIMULATIONS

Consider the Hennon map $X(t+1)=\theta(X(t))$ defined by

$$\begin{aligned}x(t+1)&=\frac{7}{5}+\frac{3}{10}y(t)-x^2(t)\\y(t+1)&=x(t).\end{aligned} \tag{22}$$

This is a famous chaotic system with a strange attractor. There are two fixed points: $\overline{X}_1\approx(0.8839,0.8839)$ with eigenvalues $\{\lambda_1^{(1)},\lambda_2^{(1)}\}=\{0.156,-1.924\}$, and $\overline{X}_2\approx(-1.5839,-1.5839)$ with eigenvalues $\{\lambda_1^{(2)},\lambda_2^{(2)}\}=\{3.26,-0.92\}$, respectively. Apparently, $\overline{X}_1$ can be stabilized through uniformly adaptive adjustment since both eigenvalues are less than unity. Since

$$\overline{\gamma}_1^{(1)}=\frac{\lambda_1^{(1)}+1}{\lambda_1^{(1)}-1}=-1.3697 \quad\text{and}\quad \overline{\gamma}_2^{(1)}=\frac{\lambda_2^{(1)}+1}{\lambda_2^{(1)}-1}=0.31601,$$

so it would be expected that the adjusted system

$$\begin{aligned}x(t+1)&=(1-\gamma)\left[\frac{7}{5}+\frac{3}{10}y(t)-x^2(t)\right]+\gamma x(t)\\y(t+1)&=(1-\gamma)x(t)+\gamma y(t)\},\end{aligned} \tag{23}$$

will converge to the fixed point $\overline{X}_1\approx(0.8839,0.8839)$ when $\gamma\in(0.31601,1)$.

Figure 4(a) shows the bifurcation diagram of $x(t)$ against the adaptive parameter γ after discarding first 300 iterations. Along with the increasing of γ the dynamics changes from pure chaos to multiple periodic points, and finally convergence to the stable fixed point $\overline{X}_1$ when $\gamma>0.3$.

To have a better idea of the effectiveness of AAM, two numerical simulations are overlapped in Fig. 4(b) for the

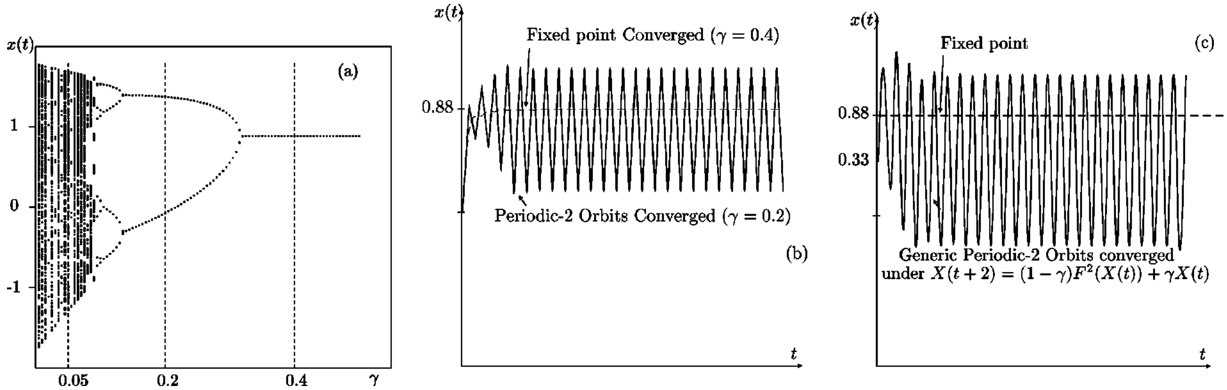

FIG. 4. Simulations for the Hennon map.

cases of $\gamma=0.2$ and $\gamma=0.4$, under which the system rapidly converges to a periodic-2 orbits and the fixed point $\overline{X}_1$, respectively. But it should be emphasized that, while the fixed point converged to, under $\gamma=0.4$ is ''generic,'' the periodic-2 orbits converged to under $\gamma=0.2$, however, is not inherited from the Hennon map.

To force an unstable nonlinear system F to converge to a generic period-m orbit, we simply need to generalize AAM to the following form:

$$X(t+1)=(1-\gamma)F^m(X(t))+\gamma X(t), \tag{24}$$

where

$$F^m \hat{=} \underbrace{F \circ F \circ \ldots \circ F}_{m\,\text{times}}$$

denotes the mth recurrent map of F.

For the Hennon map discussed, solving from the identity $\theta^2(X)=X$, a pair periodic-2 points with identical set of eigenvalues $\{-3.0101,-0.2989\}$ are obtained: $X_1^{(2)}=(1.3661,-0.6661)$ and $X_2^{(2)}=(-0.6661,1.3661)$. Since both eigenvalues are less than unity, they are easily stabilized by AAM mechanism:

$$X(t+2)=(1-\gamma)\theta^2(X(t))+\gamma X(t),$$

with $0<\gamma<1$. Computer simulation with $\gamma=0.6$ is shown in Fig. 4(c), where the system rapidly converges to ''generic'' periodic-2 points after a few iterations.

CONCLUDING REMARKS

In the implementation of AAM, only one controlling parameter γ is utilized, which depends on neither the structure of the original system nor any system parameters. A chaotic dynamical system could always be stabilized through gradually increasing γ value from zero onward, should no special requirement on any particular orbit be required.

For multidimensional systems, a more general adaptive adjustment mechanism can be designed so that every individual variable is controlled by its own adaptive parameter, that is, Eq. (2) is generalized to

$$X(t+1)=(\mathbf{I}-\Gamma)F(X(t))+\Gamma X(t), \tag{25}$$

where $\Gamma=\text{diag}\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ is a diagonal matrix. Intuitively, such generalization should be able to overcome the limitation of case III in Theorem 3 so as to stabilize any chaotic system to any desired periodic orbit by suitably choosing the Γ matrix, should *a priori* information about the structure and dynamics of the system be known in advance. While such expectation is true for a broad class of dynamical systems, there do exist some types of multidimensional systems that can never be stabilized through generalized AAM defined by Eq. (25). Detailed analysis will be followed in a forthcoming publication.

ACKNOWLEDGMENTS

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[2] In M. S. Vieira and A. J. Lichtenberg, Phys. Rev. E **54**, 1200 (1996), it is referred to as ''linear feedback algorithm.'' However, the very idea has long been applied in economics to stabilize a nonlinear system (not necessarily chaotic) in the form of ''adaptive expectation.'' See M. Nerlove, Quarterly J. Econ. **72**, 227 (1958), R. A. Heiner, J. Econ. Behav. & Org. **12**, 233 (1989), and a detailed discussion in W. Huang, Working paper, Howard University, 1992 (unpublished).

[3] AAM with $\gamma>1$ should be implemented with caution, because it actually destabilizes all original stable periodic orbits.